Examinee’s No.___________________

Department of Systems Innovation /
Department of Nuclear Engineering and Management /
Department of Technology Management for Innovation,
Graduate School of Engineering, The University of Tokyo
2015 Entrance Examination Test and Answer Sheets

Mathematical Problems Designed to Test Ability to Think Logically

Monday, August 25, 2014   13:00 - 15:30

Notice:

1. Do not open this booklet until the start of the examination.

2. If you find missing or badly printed pages, please notify the supervisor.

3. Write your examinee number at top of this page. Do not write your examinee‘s number or name anywhere else in this booklet.

4. Write the answers including the outline of your solutions below each question.

5. Blank sheets for calculation are distributed separately. After the examination, they must be returned.

6. This test has 20 questions in total. Choose and answer 15 questions out of 20.
<u>Check the question numbers of the questions you have selected in the table below. Please note that you are not allowed to answer more than 15 questions.</u>

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Q. 1

Find the determinant of the following matrix.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a & a^2 & a^3 & a^4 \\ b & b^2 & b^3 & b^4 \\ c & c^2 & c^3 & c^4 \end{vmatrix}$$

Q. 2

Let S be the triangle that is part of the plane $2x+2y+z=2$ that lies in the first octant. Let $\boldsymbol{A}$ be the vector field $\boldsymbol{A}=x\boldsymbol{i}-2z\boldsymbol{k}$. Find the surface integral of $\boldsymbol{A}$ over S, where $\boldsymbol{i}=(1,0,0)$ and $\boldsymbol{k}=(0,0,1)$.

Q. 3

Obtain the equation for the trajectory of the point $P(x, y)$, where x and y are functions of the parameter t.

$$\begin{cases} x = \dfrac{1 - t^2}{1 + t^2} \\ y = \dfrac{4t}{1 + t^2} \end{cases}$$

Q. 4

Show that $\boldsymbol{x}^T A\boldsymbol{x} > 0$ with arbitrary non-zero vector $\boldsymbol{x}$ for the following matrix A.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Q. 5

There is a triangular prism with infinite height. It has three edges parallel to z-axis, each passing through points (0,0,0), (3,0,0) and (2,1,0) respectively. Calculate the volume within its side surfaces as well as the planes $z = 2x + 3y + 6$ and $z = 2x + 7y + 8$.

Q. 6

For a matrix A (see below),

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(1) find eigenvalues of A;

(2) prove that $A^n = A^{n\text{-}2} + A^2 - I \quad (n \geqq 3)$, where I is an identity matrix;

(3) calculate A^{200}.

Q. 7

The z-plane $z = x + iy$ is mapped to the w-plane $w = u + iv$ (where $w = \frac{1}{z+1}$). Find the image projected onto the complex w-plane from the line $y = x$ on the complex z-plane.

Q. 8

The given series has 15 terms:

1+1, 2+3+4+9, 8+27+16+81+32+243, …

Find the sum of the terms of this series.

Q. 9

Note that the cipher text “MMZ-MAZ-MMA-ZMM-MAZ-MMA” stands for “L-O-N-D-O-N”, whereas “ZZA-ZMA-AZZ-MMZ-MZZ-MMA” stands for “B-E-R-L-I-N”.

Encrypt “P-A-R-I-S” by applying the rule used in the cipher texts above.

Q. 10

(1) Consider a line in a 3-dimensional space expressed as $\boldsymbol{x} = \boldsymbol{a}t$, where $\boldsymbol{a}$ is a 3×1 vector and t is an arbitrary real number. The point $\boldsymbol{p}$ is the projection of point $\boldsymbol{b}$ onto that line (i.e. the nearest point on that line). Express $\boldsymbol{p}$ as a function of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(2) Consider a plane in a 3-dimensional space expressed as $\boldsymbol{x} = A\boldsymbol{s}$, where A is a 3×2 matrix and $\boldsymbol{s}$ is an arbitrary 2×1 vector. The point $\boldsymbol{p}$ is the projection of point $\boldsymbol{b}$ onto that plane (i.e. the nearest point on that plane). Express $\boldsymbol{p}$ as a function of A and $\boldsymbol{b}$.

Q. 11

Suppose that a solution of the following differential equation,

$$\frac{d^2 f}{dr^2} + \left[1 - \frac{2}{r} - \frac{k(k+1)}{r^2}\right] f = 0 \quad (r \geqq 0)$$

where k is an integer not smaller than 2, has Taylor series,

$$f(r) = \sum_{n=1}^{\infty} c_n r^n$$

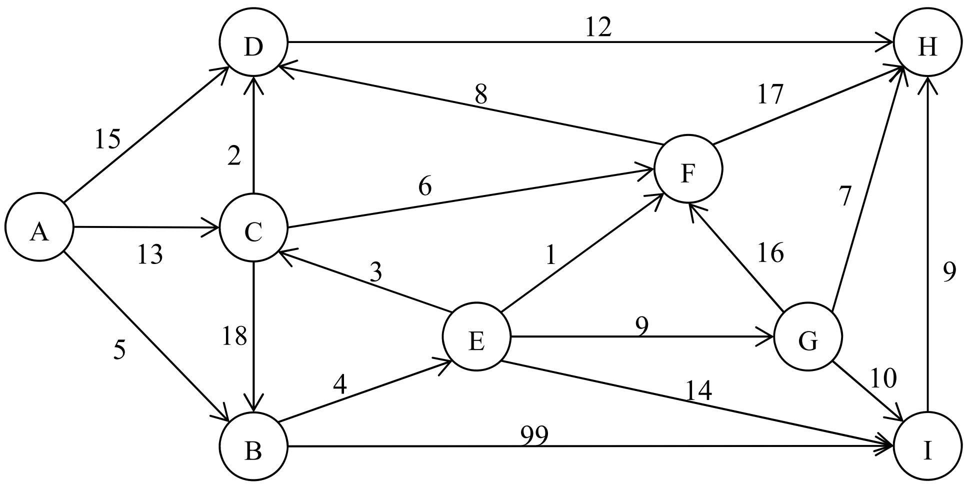
with $c_{k+1} = 1$ at $r = 0$.

(1) Find c_k.

(2) Find c_{k+2}.

Q. 12

In the directed graph shown below, find the distance of the shortest path from vertex A to each and every vertex, and indicate the vertex prior to the last one on each path. Write your answer in the table below. Note that the numbers beside the edges indicate the distance.



Vertex	Distance from A	Vertex prior to the last one
B	5	A
C		
D		
E		
F		
G		
H		
I		

Q. 13

Using a certain mathematical rule, several integers are arranged as shown below. Substitute “A” with an appropriate integer, and explain the rule being applied.

			A			
		57		54		
	73		94		63	
83		77		59		36

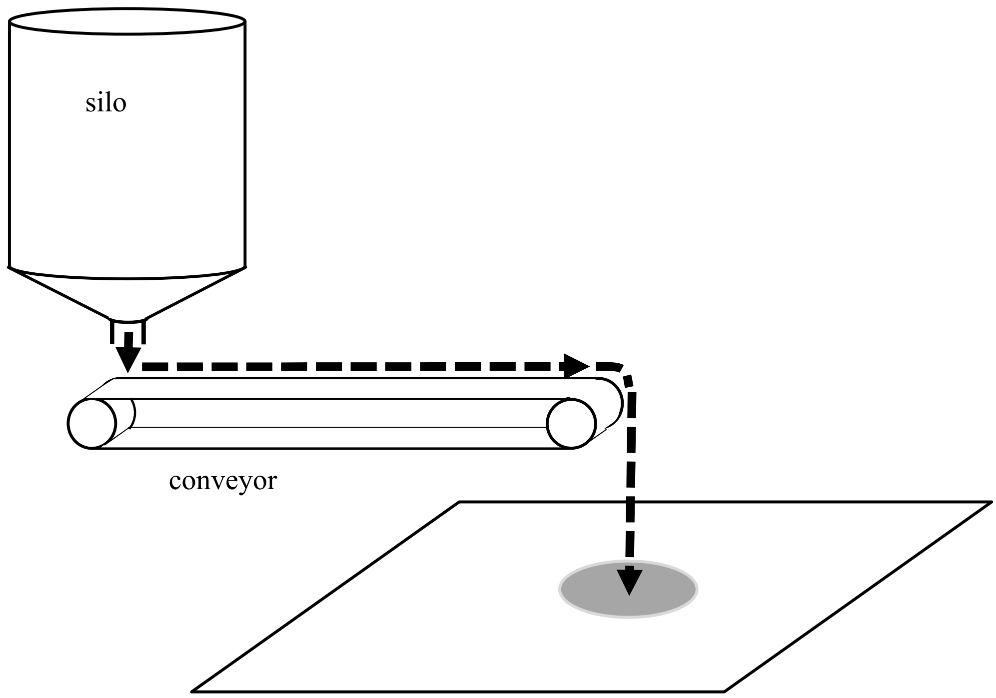
Q. 14

The number 153 has an interesting property. In other words, it is equal to the sum of the cubes of its digits: $1^3+5^3+3^3=153$.

370 is another number with the same property. There are two other three-digit natural numbers with three different digits, both smaller than 500, that have the same property. Find them.

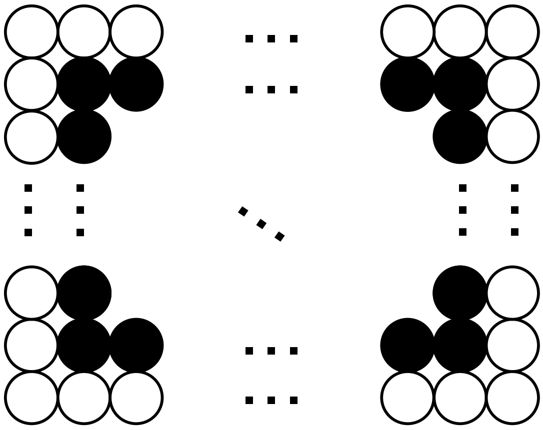
Q. 15

Wheat carried from a silo by a conveyor piles up, maintaining a circular conic shape in which the height from its base is always one-half of the radius of the base. The wheat is being poured to the pile at 12 m^3/min. Find the rate (in m/min) at which the height of the pile increases, when the height was 3 m. Express your answer in terms of π.



Q. 16

Black and white stones are arranged, as shown below. Black stones are placed within a rectangle, whereas the white stones are placed around that rectangle. It turned out that the number of white stones is double the number of black stones. Find the numbers of each type of stones.



Q. 17

Several thieves (A, B, C, …), who are members of a group, attacked a jewellery shop and stole 72 diamonds of the same value. First, thief A proposes how to distribute the 72 diamonds to all thieves. His/her proposal would be accepted if at least the half of all the members (including the proposer himself/herself) supports it. However, if the proposal is rejected by more than the half of the members, the thief A would be expelled from the group, and thus the thief B would make another proposal to all remaining members. If the proposal made by the thief B is not accepted, he/she would also be expelled, and then the thief C would make a new proposal, and so on and so forth. When making a proposal, or when supporting or rejecting others' proposals, each thief aims to receive as many diamonds as possible while avoiding being expelled. In addition, a thief randomly supports or rejects a proposal, if his/her decision does not affect the number of the diamonds he/she receives, as long as he/she remains in the group. Furthermore, each thief knows that the other thieves follow the same rules as above. If a thief is expelled, he/she will receive no diamonds.

(1) Consider a case where there are only 3 thieves (i.e. A, B, and C) in the group. Find how many diamonds thief A can receive at most. In addition, give the details of the proposal that the thief A made to other thieves at that time.

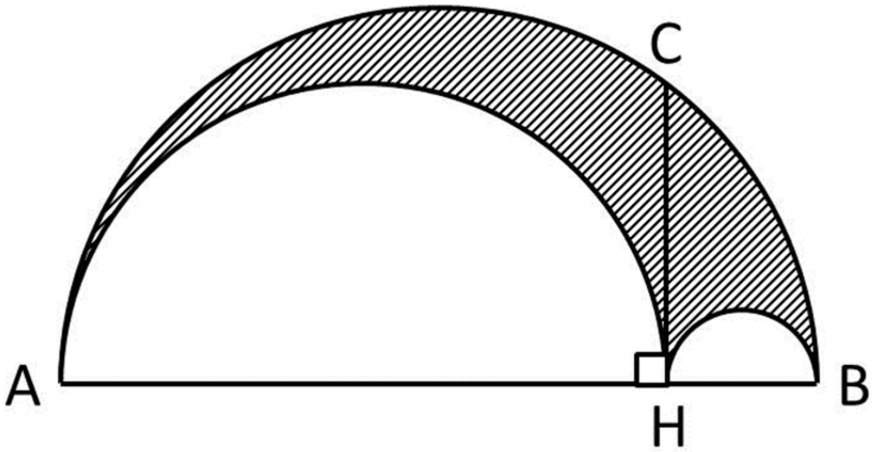
(2) Consider another case where there are only 9 thieves (i.e. A, B, C, D, E, F, G, H, and I) in the group. Find how many diamonds thief A can receive at most. In addition, give the details of the proposal that the thief A made to other thieves at that time.

Q. 18

Suppose you have three one-yen coins and four ten-yen coins in your right pocket, as well as two one-yen coins and a fifty-yen coin in your left pocket. You choose one of the two pockets at random, and select a coin from it at random. If the selected coin is a one-yen coin, what is the probability that it came from the right pocket?

Q. 19

There are three semicircles (see the figure below). Calculate the surface of the shaded area, if the length of the line segment CH is 4 cm. Note that CH is perpendicular to the line segment AB. Express your answer in terms of π.



Q. 20

A bar has seven seats: ①, ②, ③, ④, ⑤, ⑥, and ⑦, as shown in the figure below.

①	②	③	④	⑤	⑥	⑦

Suppose that seven customers enter the bar one after the other. Each of them takes a seat until all seats are occupied. Each customer chooses his/her seat with an equal probability. If possible, he/she does not take a seat next to any already occupied seat.

If neither seat to his/hers is occupied when a customer takes a seat, he/she is "satisfied". On the other hand, a customer who takes a seat next to any already occupied seat is "unsatisfied".

(1) Find how many orders can be taken in total, if the number of the "satisfied" customers exceeds the number of the "unsatisfied" ones.

(2) Find how many orders can be taken in total.

(3) If the first customer is invited to take the seat ①, calculate the increase of the mean number of the "satisfied" customers.